Diagrams of the linear Backlund transformation of the cylindrical tow-dimensional Toda lattice

1989 J. Phys. A: Math. Gen. 22 5017

(http://iopscience.iop.org/0305-4470/22/22/026)

J. Phys. A: Math. Gen. **22** (1989) 5017–5032. Printed in the UK

Diagrams of the linear Bäcklund transformation of the cylindrical two-dimensional Toda lattice

Fukuko Yuasa† and Noriko Saitoh‡

† National Laboratory For High Energy Physics (KEK), Oho, Tsukuba, Ibaraki 305, Japan
‡ Department of Applied Mathematics, Faculty of Engineering, Yokohama National University, Hodogaya-ku, Yokohama 240, Japan

Received 6 March 1989

Abstract. A fundamental set of solutions to the linear Bäcklund transformation (LBT) is obtained in a general form in the cylindrically symmetric Toda lattice. Diagrams of solutions generated by the LBT are drawn and their local behaviours are entirely known once only one solution is given.

1. Introduction

Since the Toda lattice was discovered twenty years ago [1], it has played a crucial role in the study of nonlinear physics [2]. Among various works on this model the generalisation to the two-dimensional Toda lattice (2DTL) has many interesting features and has been studied in rather more detail [3, 4]. First of all it offers a simple but non-trivial integrable model in three-dimensional spacetime [3].

Besides the well known soliton solutions there have been found many solutions to the 2DTL. For example Nakamura obtained cylindrically symmetric solutions [5] expressed by Bessel functions in analogy with the case of the two-dimensional KdV equation [6]. His approach is based on the use of the Bäcklund transformations discovered by Hirota [7] which have proven quite useful. Kametaka found other types of Bäcklund transformations [8] which enabled him to solve Euler–Poisson–Darboux types of linear differential equations to obtain solutions to the 2DTL with rational and hypergeometric forms. These studies are very interesting in themselves and should be quite fruitful when we investigate the 2DTL in its application to real physical processes.

Following Nakamura's investigation of the 2DTL the authors studied a pair of generalised recurrence formulae (GRF) [9] to which Bessel functions follow as a special case. The 2DTL can be derived as a compatibility condition [9, 10] of this pair of linear equations. Writing them explicitly we find them to be equivalent to the Bäcklund transformations by Hirota [11] which are mentioned above.

Some detailed studies of the GRF revealed that it has a gauge symmetry and a symmetry under the exchange of the roles of the field amplitude and the gauge field [12–14], which we called the dual symmetry. Namely we can write the GRF in a gauge-invariant form. When we fix the gauge appropriately the GRF becomes dual symmetric. This property of the GRF is remarkable. When the gauge field, say g_n, satisfies the 2DTL as a compatibility condition, the same is true for the field amplitude, say f_n. If g_n is a solution of the 2DTL and the compatibility condition is satisfied, the

GRF is reduced to a single linear differential equation of f_n whose coefficients are fixed by g_n. Then, owing to the duality of GRF, f_n itself is a solution to the 2DTL. Therefore the GRF provides a scheme of the Bäcklund transformation in the form of a linear differential equation, which we will call the linear Bäcklund transformation, or the LBT for short. This scheme enables us to obtain solutions of the 2DTL [15, 17, 18], merely by solving the LBT.

The LBT is a second-order partial differential equation which can be solved iteratively to give unique and bounded solutions under certain boundary conditions [15]. Especially in the cylindrically symmetric case, the LBT becomes an ordinary differential equation [15]. The ordinary soliton solution in multidimensional spacetime is essentially a one-dimensional excitation whose dependence on the variables is the same as in the case of one dimension. On the other hand, Nakamura's cylindrically symmetric solutions are essentially two-dimensional excitations whose topology is not Euclidean. From the physical point of view, we are interested in excitations having an intrinsic dependence on higher dimensions, such as vortices, waves propagating in a radial direction, and so on. These excitations should be considered as solutions of equations of motion on non-Cartesian coordinates. For example, radially propagating ion acoustic waves in a collisionless plasma have been considered by Maxon and Viecelli [16] using the modified KdV equation.

From these considerations, we concern ourselves, in this paper, only with the case where the system keeps cylindrical symmetry. The above procedure of the LBT was also carried out in the cylindrically symmetric case to find a series of solutions of the 2DTL in our previous papers [15, 19].

The main purpose of this paper is to present the general form of a fundamental set of solutions of the LBT so as to keep the dual symmetry, and to draw a diagram of solutions generated by the LBT. The dual symmetry becomes obscured in the cylindrically symmetric case. But it will be shown that it still plays a central role in generating new solutions.

This paper is organised as follows. In § 2, we present the gauge-symmetric formulation, the duality relation and the LBT of the cylindrically symmetric 2DTL, and develop a systematic way to get successive solutions. Following the results of § 2, we deduce a unit triangle diagram of solutions in § 3. This triangle consists of a particular solution of the 2DTL and a fundamental pair of solutions of the LBT generated by it. One of the pair will be obtained immediately from the duality relation. In § 4, we construct an entire diagram of the solutions of cylindrically symmetric 2DTL, which is the main purpose of this paper. In § 5, we discuss local behaviours of the solutions generated by the LBT making use of the study of linear differential equations and present some examples. The final section is devoted to summaries and discussions.

2. Gauge-symmetric formulation of cylindrically symmetric 2DTL

It has already been shown that the 2DTL system as well as the one-dimensional Toda lattice system has a dual symmetry between field amplitudes and gauge fields. In other words, the 2DTL keeps the 'duality relation', which means that the field and gauge field can exchange their roles leaving the expression of the GRF, and therefore of their compatibility condition, unchanged. By means of the duality relation, one can carry out the following procedures repeatedly to obtain solutions of the nonlinear 2DTL equation. First, once a gauge field is fixed to be a solution of the 2DTL, one solves the

LBT and obtains an analytical expression of a field amplitude of the 2DTL. Second, we regard the solution obtained by solving the LBT as a gauge field, and solve the LBT to obtain a successive solution.

In the cylindrically symmetric case [15], the dual symmetry is not manifest. We are, however, able to find a way to recover this important relation. Now, we shall review briefly the dual symmetry in the cylindrically symmetric case.

Let us start from the bilinear cylindrical 2DTL equation [5]:

$$f_n\left(\partial_\rho^2+\frac{1}{\rho}\partial_\rho\right)f_n-(\partial_\rho f_n)^2-(f_{n+1}f_{n-1}-f_n^2)=0 \tag{1}$$

with definitions

$$\rho=\sqrt{x^2+y^2} \qquad \partial_\rho=\partial/\partial_\rho \qquad f_n=f_n(\rho).$$

The duality equations for f_n appear in the form

$$f_{n+1}=-\frac{g_{n+1}}{g_n}\left(\partial_\rho-\frac{b_n}{\rho}-\partial_\rho \ln g_{n+1}\right)f_n \tag{2a}$$

$$f_{n-1}=\frac{g_n}{g_{n+1}}\left(\partial_\rho-\frac{a_n}{\rho}-\partial_\rho \ln g_n\right)f_n \tag{2b}$$

where the parameters a_n and b_n depend merely on the discrete variable n indicating a lattice site. The nonlinear equation (1) is linearised into

$$\partial_\rho^2 f_n+\left(\frac{1-(a_n+b_n)}{\rho}-\partial_\rho(\ln g_n g_{n+1})\right)(\partial_\rho f_n)$$
$$+\left[1+\left(\frac{a_n}{\rho}+\partial_\rho \ln g_n\right)\left(\frac{b_n}{\rho}+\partial_\rho \ln g_{n+1}\right)\right]f_n=0 \tag{3}$$

where g_n is a solution of the 2DTL.

From the compatibility condition of (2a) and (2b), a_n and b_n are determined as [15]

$$a_n=-n+c' \tag{4}$$

and

$$b_n=n-c'-c \tag{5}$$

with arbitrary constants c' and c, and g_n is determined so as to satisfy (1), i.e.

$$g_n\left(\partial_\rho^2+\frac{1}{\rho}\partial_\rho\right)g_n-(\partial_\rho g_n)^2-(g_{n+1}g_{n-1}-g_n^2)=0. \tag{6}$$

Accordingly, if a solution of (6) is found, substituting it into (3) as g_n, one can solve (6) and generate a successive solution.

The duality equations (2a, b) can be rewritten into the duality equations for g_{n+1}:

$$g_{n+2}=-\frac{f_{n+1}}{f_n}\left(\partial_\rho-\frac{-a_{n+1}}{\rho}-\partial_\rho \ln f_{n+1}\right)g_{n+1} \tag{7a}$$

$$g_n=\frac{f_n}{f_{n+1}}\left(\partial_\rho-\frac{-b_n}{\rho}-\partial_\rho \ln f_n\right)g_n. \tag{7b}$$

Comparing (7*a*, *b*) with (2*a*, *b*), we notice that the roles of g_n and f_n are exchanged and, with replacement of $-a_{n+1}$ in (7*a*) by b_n and $-b_n$ in (7*b*) by a_n, two pairs of duality equations have the same expression. By means of the duality equations (7*a*, *b*), the bilinear 2DTL equation for g_{n+1} is linearised to a linear differential equation, i.e. the LBT for g_{n+1}, with contains the already known solution f_n in its coefficients:

$$\partial_\rho^2 g_{n+1} + \left(\frac{1-(-b_n - a_{n+1})}{\rho} - \partial_\rho(\ln f_n f_{n+1})\right)(\partial_\rho g_{n+1})$$
$$+\left[1+\left(\frac{-b_n}{\rho}+\partial_\rho \ln f_n\right)\left(\frac{-a_{n+1}}{\rho}+\partial_\rho \ln f_{n+1}\right)\right] g_{n+1} = 0. \tag{3'}$$

When the duality equations are rewritten repeatedly, the set of parameters in the duality equations change subsequently as follows:

$$a_n \to -b_n \to a_{n+1} \to -b_{n+1} \to a_{n+2} \ldots$$
$$b_n \to -a_{n+1} \to b_{n+1} \to -a_{n+2} \to b_{n+2} \ldots .$$

On the other hand, the sequence of solutions generated by the LBT is

$$\begin{aligned} &g_n \to f_n \\ &\quad f_n \to g_{n+1} \\ &\qquad g_{n+1} \to f_{n+1} \\ &\qquad\qquad \ldots . \end{aligned}$$

From these two schemata we can draw the important result that, by choosing the set of parameters appropriately, g_{n+1} is a solution of the LBT generated by f_n if f_n itself is generated by g_n.

3. Fundamental set of solutions to the LBT and a unit triangle

In the cylindrically symmetric case, the LBT is an ordinary second-order differential equation and therefore it has a fundamental set of solutions. In § 2, it was shown that the duality relation always gives one of the solutions to the LBT. In this section, we will derive the other solution and present a general form of fundamental set of solutions to the LBT.

The LBT can be written [5] as

$$\partial_\rho^2[{}_kf_n] + {}_kP_n\partial_\rho[{}_kf_n] + {}_kQ_n[{}_kf_n] = 0 \qquad k \geq 1 \tag{8}$$

where the coefficients are given by a known solution of the 2DTL in the following forms:

$${}_kP_n \equiv -\partial_\rho \ln(\rho^{a_n+b_n-1}\, {}_{k-1}g_n \cdot {}_{k-1}g_{n+1}) \tag{9}$$

$${}_kQ_n \equiv 1 + [\partial_\rho \ln(\rho^{a_n}\, {}_{k-1}g_n)][\partial_\rho \ln(\rho^{b_n}\, {}_{k-1}g_{n+1})]. \tag{10}$$

Here k indicates the number of repetition of the LBT. If we know a solution of (8), say ${}_kf_n$, another independent solution can be obtained, by the standard method, in the form

$$\begin{aligned} {}_kf'_n &= {}_kf_n \int^\rho \frac{\exp(-\int^{\rho'} {}_kP_n \, \mathrm{d}\rho'')}{({}_kf_n)^2}\,\mathrm{d}\rho' \\ &= {}_kf_n \int^\rho \frac{{}_{k-1}g_n \cdot {}_{k-1}g_{n+1}}{\rho^{1-(a_n+b_n)}({}_kf_n)^2}\,\mathrm{d}\rho'. \end{aligned} \tag{11}$$

From the argument of the previous section it is clear, owing to the duality relation, that ${}_{k-2}f_{n+1}$ is one of the solutions generated by the gauge field ${}_{k-1}g_n$ if ${}_{k-1}g_n$ itself is generated by ${}_{k-2}f_n$. We can choose ${}_kf_n$ for this solution without loss of generality, i.e.

$$ {}_kf_n = {}_{k-2}f_{n+1}. \tag{12} $$

Then the other solution is given by

$$ {}_kf_n' = {}_{k-2}f_{n+1} \int^{\rho} \frac{{}_{k-1}g_n \cdot {}_{k-1}g_{n+1}}{\rho^{1-(a_n+b_n)}({}_{k-2}f_{n+1})^2}\, \mathrm{d}\rho'. \tag{13} $$

Since the Wronskian of these two solutions (12) and (13) is given by

$$ W = \exp\left(-\int^{\rho} {}_kP_n\, \mathrm{d}\rho'\right) = \frac{{}_{k-1}g_n \cdot {}_{k-1}g_{n+1}}{\rho^{1-(a_n+b_n)}}\bigg|_{\rho_0}^{\rho} \tag{14} $$

they are independent unless ${}_{k-1}g_n$ is identically zero and constitutes a fundamental set of solutions of (8).

Before going into details of drawing a general diagram we summarise the results we have found so far in a schematic form in figure 1. The fundamental constitutions of this diagram are the duality relation (DR) and the LBT which are indicated by broken and full lines respectively. The horizontal line of the triangle with two arrows indicates a linearly independent pair of solutions.

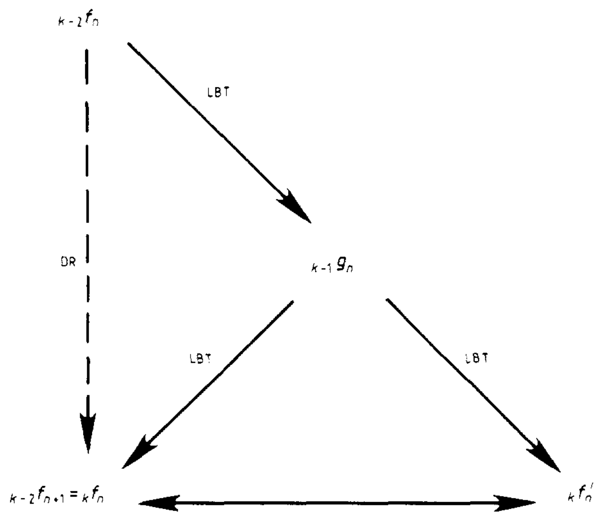

Figure 1. Unit triangle. Full lines represent generation of solutions by the LBT and broken lines the DR.

4. Diagrams of the LBT

In this section we consider successive generation of solutions based on the LBT which we discussed in the former sections. As was shown in § 2 the duality relation offers a solution generated by a given gauge (potential) field along the vertical direction. Once this solution is given, the argument of § 3 enables us to find another solution in the horizontal direction which is linearly independent but shares the same generation. Combining them together we can generate an infinite family of solutions by simply applying these procedures at each generation.

Now let us assume that a solution of the 2DTL is known, say ${}_0g_n$, and the LBT (8) is solved to obtain a solution, say ${}_1f_n$. We denote another solution of (8), linearly independent of ${}_1f_n$, by ${}_1\tilde{f}_n$, taking account of the fact that most of the solutions of the kth generation are given by those of the $(k-2)$th generation, i.e.

$$ {}_kf_n = {}_{k-2}f_{n+1} $$

$$ {}_kg_n = {}_{k-2}g_{n+1} . $$

We draw a diagram of solutions generated by the LBT in figure 2.

Note that all elements of the kth generation are written in the kth row of this diagram. We also recall that the sets of parameters (a_n, b_n) and $(-b_n, -a_{n+1})$ must be used as we derive new solutions depending on either (3) or (3′) corresponding to an even or odd row of generation as indicated in the diagram. The most significant feature of this diagram is that the truly new solutions are those lying along the slanting lines on the extreme right and left which are indicated by bold arrows. From the construction it is clear that the family of solutions generated along the right extreme line is quite independent of the one in the left extreme. The former series can be obtained by integrations of functions containing ${}_0f_n$ only, whereas the latter contains ${}_0f_n$ only.

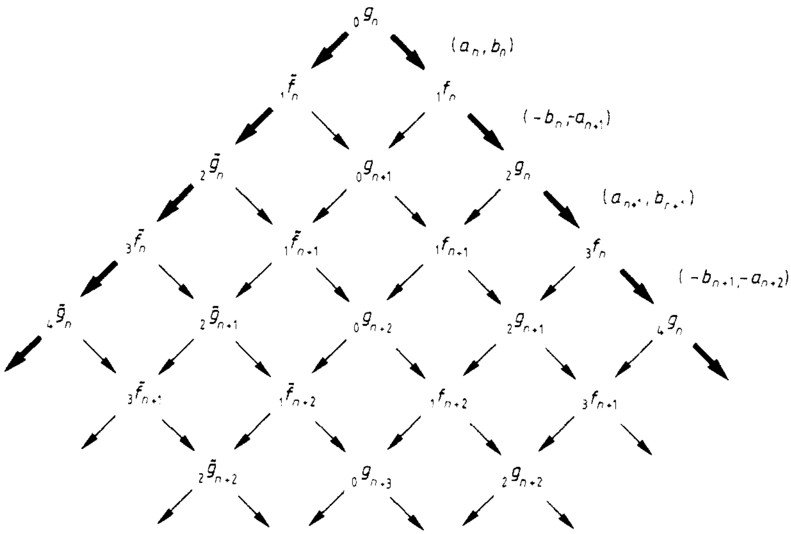

Figure 2. Diagram of solutions by the LBT. New solutions appear in both sides of the diagram along the bold arrows when we choose sets of parameters as round brackets.

For illustration let us choose the simplest solution 1 for ${}_0g_n$ and set $a_n = -b_n = -n$. Then we obtain immediately

$$ {}_1f_n = J_n(\rho) \qquad \text{and} \qquad {}_1\tilde{f}_n = N_n(\rho) $$

where J_n and N_n are Bessel and Neumann functions of order n respectively. The corresponding diagram is presented in figure 3. It is very clear in figure 3 that the

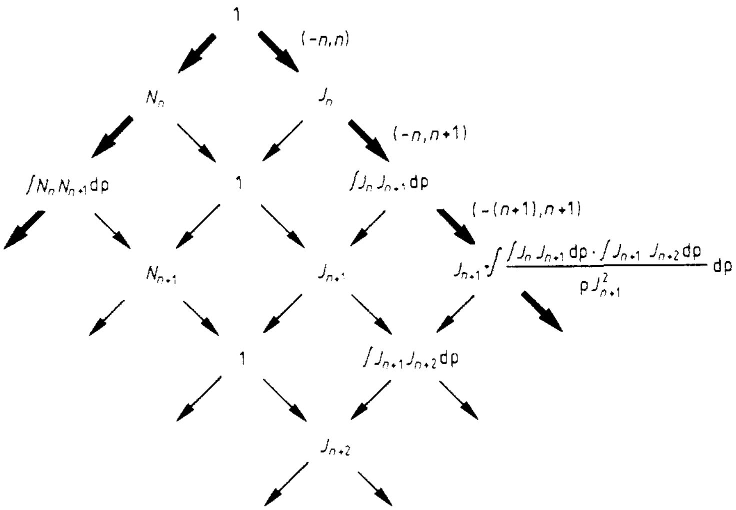

Figure 3. Examples of solutions by the LBT. Bessel-type solutions are obtained if we start the LBT from a trivial solution, say 1, and choose sets of parameters as round brackets.

series of J_n and N_n are uncorrelated from each other. Some solutions indicated here explicitly are those already given in Nakamura's work [5] and in our previous papers [18, 19].

5. Behaviour of solutions generated by the LBT and DR

In the previous sections we have shown a method of drawing a net diagram of solutions of the 2DTL by means of the LBT and the DR. This method is very simple and systematic because after finding the first triangle $({}_0g_n, {}_1f_n, {}_1\tilde{f}_n)$, we can derive the second generation of triangles $({}_1f_n, {}_0g_{n+1}, {}_2g_n)$ and $({}_1\tilde{f}_n, {}_0g_{n-1}, {}_2\tilde{g}_n)$ merely by using the formula (11) and the DR. It is a very important and useful fact that the DR guarantees that ${}_0g_{n+1}$, whose analytic feature is known, can be a solution of the LBT. This will become clear in the following arguments, in which we will discuss the behaviours of solutions generated by the LBT and the DR.

5.1. General aspects

Nakamura has already pointed out [5] that a Bessel-type solution J_n is a divergent one whereas $1+\varepsilon^2\sum_{m=n}^{\infty} J_m^2$ gives a finite one as they are transformed into physical amplitudes of the cylindrically symmetric case of the 2DTL system, which is the same situation as the cylindrically symmetric KdV equation [6]. This conclusion is based on the fact that the latter is larger than 1 at every point of the value of dependent variable whereas the former has zero points. These properties will be also found in the series of solutions which are obtained by the LBT and the DR. We can analyse these properties systematically based on the method of studying linear equations.

First of all, let us study the local behaviour of solutions around $\rho=0$. In the unit triangle of the diagram of the kth LBT, let us assume that the potential is expanded in a power series of ρ. To be specific we assume the kth LBT equation is generated by ${}_{k-1}g_n$ and solved for ${}_kf_n$. Then we expand ${}_{k-1}g_n$ in the following form:

$$ {}_{k-1}g_n(\rho) = \rho^{{}_{k-1}s_n} G_n(\rho) \tag{15}$$

with the index ${}_{k-1}s_n$ indicating the lowest order of expansions around $\rho = 0$ and the analytic function $G_n(\rho)$ at $\rho = 0$ whose first term is non-zero valued:

$$ G_n(\rho) = G_{n0} + G_{n1}\rho + G_{n2}\rho^2 + \dots . \tag{16}$$

As in the general case of ordinary linear equations, analytic properties of solutions to the LBT (8) are governed by the coefficients ${}_kP_n$ and ${}_kQ_n$, which are given by (10). According to the expressions (15) and (16), we get the following expressions for ${}_kP_n$ and ${}_kQ_n$ around $\rho = 0$:

$$ {}_kP_n(\rho) = -\frac{{}_ka_n + {}_kb_n - 1 + {}_{k-1}s_n + {}_{k-1}\bar{s}_n}{\rho} - \frac{(G_nG_{n+1})'}{G_nG_{n+1}} \tag{17}$$

$$ {}_kQ_n(\rho) = 1 + \frac{({}_ka_n + {}_{k-1}s_n)({}_kb_n + {}_{k-1}\bar{s}_n)}{\rho^2} + \frac{{}_ka_n + {}_{k-1}s_n}{\rho}\frac{G'_{n+1}}{G_{n+1}} + \frac{{}_ka_n + {}_{k-1}\bar{s}_n}{\rho}\frac{G'_n}{G_n} + \frac{G'_nG'_{n+1}}{G_nG_{n+1}} \tag{18}$$

in which we denote ${}_{k-1}\bar{s}_n$ with the meaning

$$ {}_{k-1}\bar{s}_n \equiv {}_{k-1}s_{n+1} . $$

From the construction of the coefficients of ${}_kP_n$ and ${}_kQ_n$, $\rho = 0$ appears to be a regular singular point of the kth LBT equation. Then we can get at least one power series solution which converges and has no logarithmic singularity around $\rho = 0$, following the usual method of power expansion. To obtain this solution, let us expand the solution of the kth LBT equation as

$$ {}_kf_n(\rho) = \rho^{{}_ks_n} \sum_{j=0}^{\infty} c_j^k \rho^j \tag{19}$$

with a non-zero coefficient c_0^k. Inserting expansions (17), (18) and (19) to the LBT equation, we get the indicial equation for the indices of power series solutions in the form

$$ ({}_ks_n - {}_ka_n - {}_{k-1}s_n)({}_ks_n - {}_kb_n - {}_{k-1}s_{n+1}) = 0. \tag{20}$$

Then the solution which converges and has no logarithmic singularity at $\rho = 0$ is determined to have the index

$$ {}_ks_n = \max({}_ka_n + {}_{k-1}s_n, {}_kb_n + {}_{k-1}s_{n+1}). \tag{21}$$

Furthermore, if we assume that the potential ${}_{k-1}g_n$ is either an even or odd function of ρ, only c_j^k of even j in the expansion (19) are related through the kth LBT equation. We can choose the coefficient c_1^k to be zero from the equation for the power of $\rho^{{}_ks_n - 1}$. Hence the summation in (19) is an even function and the even-odd property of the solution ${}_kf_n$ is uniquely determined by ${}_ks_n$ in this case.

Another power series solution linearly independent of the former solution with the index (21) generally has a logarithmic singularity at $\rho = 0$ when the difference of two indices is an integer.

A detailed examination of the coefficients of the LBT is required to determine whether the solution with smaller index has a logarithmic singularity or not. However, we would like to show in the following that a careful study of the diagram of the LBT and the DR enables us to conclude that, under certain conditions, all of the solutions have no such singularities at $\rho = 0$.

From the indicial equation (20) we have two indices given by

$$\begin{aligned} {}_k s_n^{(1)} &= {}_k a_n + {}_{k-1} s_n \\ {}_k s_n^{(2)} &= {}_k b_n + {}_{k-1} s_n . \end{aligned} \tag{22}$$

We have expressions of ${}_k a_n$ and ${}_k b_n$ corresponding to even and odd values as

$$\begin{array}{lll} {}_{2l} a_n = a_{n+l-1} + \gamma & {}_{2l} b_n = -a_{n+l} & l \geqslant 1 \\ {}_{2l+1} a_n = a_{n+l} & {}_{2l+1} b_n = -a_{n+l} - \gamma & l \geqslant 0 \end{array}$$

where we write ${}_1 a_{n+l} = a_{n+l}$ and so on, following the expression of § 2. Using (4) and (5) we get

$$\begin{array}{ll} {}_{2l} a_n = -n + c - l + 1 + \gamma & {}_{2l} b_n = n - c + l \end{array}$$

and (23)

$$\begin{array}{ll} {}_{2l+1} a_n = -n + c - l & {}_{2l+1} b_n = n - c + l - \gamma \end{array}$$

with arbitrary constants γ and c.

For a given k there are two solutions characterised by the indices ${}_k s_n^{(1)}$ and ${}_k s_n^{(2)}$ of (22). Their values are determined by those of $(k-1)$th level. In general, we do not know whether the indices of the potential, ${}_{k-1} s_n$ in (22) are the type of ${}_{k-1} s_n^{(1)}$ or ${}_{k-1} s_n^{(2)}$. For the diagram of the LBT and the DR to be generated consistently, however, they must satisfy

$${}_{k+1} s_n = {}_{k-1} s_{n+1} . \tag{24}$$

Let us assume at this stage that (22) is satisfied by

$${}_k s_n^{(1)} = {}_k a_n + {}_{k-1} s_n^{(2)} \qquad {}_k s_n^{(2)} = {}_k b_n + {}_{k-1} s_{n+1}^{(2)} \tag{22$'$}$$

for all k. Then it is straightforward to see that (24) is satisfied by

$${}_{k+1} s_n^{(1)} = {}_{k-1} s_{n+1}^{(2)} . \tag{24$'$}$$

Namely, the particular set of indices (22′) is consistent with the diagram. In this case we can calculate all values of the indices from (23), (22′) and (24′) and obtain

$${}_k s_n^{(1)} = (k-2)(n-c) + \begin{pmatrix} \frac{1}{4}(3k^2 - 8k + 7) - \dfrac{k-1}{2}\gamma \\ \frac{1}{4}(3k^2 - 10k + 8) - \dfrac{k-2}{2}\gamma \end{pmatrix} + {}_0 s_{n+k-1} \tag{25a}$$

and

$${}_k s_n^{(2)} = k(n-c-1) + \begin{pmatrix} \frac{1}{4}(3k^2 + 2k - 1) - \dfrac{k+1}{2}\gamma \\ \frac{1}{4}(3k^2 + 2k) - \dfrac{k}{2}\gamma \end{pmatrix} + {}_0 s_{n+k} \tag{25b}$$

where the upper and the lower values in round brackets correspond to odd and even values of k, respectively. The difference of (25a) and (25b) is given by

$$ {}_ks_n^{(1)} - {}_ks_n^{(2)} = -2k-2n+2c+2+\gamma+{}_0s_{n+k-1} = {}_0s_{n+k}. \tag{26}$$

Without loss of generality we can number the lattice such that n is always larger than $c+1+\gamma/2$. Then the difference (26) is negative definite as long as ${}_0s_n^{(2)}$ is an increasing function on n. In other words, under the condition that the index ${}_0s_n^{(2)}$ of the function introduced at the beginning of series of the Bäcklund transformations is an increasing function of n, ${}_ks_n^{(1)}$ is smaller than ${}_ks_n^{(2)}$ on each level of k.

Now let us look at the diagram in figure 2, and suppose that the index ${}_1s_n^{(2)}$ corresponds to ${}_1f_n$ and the above condition is satisfied for ${}_0g_n$. Then ${}_1f_n$ converges and has no logarithmic singularity at $\rho=0$. The index ${}_3s_n$ of ${}_3f_n={}_1f_{n+1}$ is given by ${}_1s_{n+1}^{(2)}$, because of the DR, which is also equal to ${}_3s_n^{(1)}$ due to (24′). This implies that the index of ${}_3f_n$ must be ${}_3s_n^{(2)}$ which is larger than ${}_3s_n^{(1)}$. Therefore ${}_3f_n$ is also convergent and has no logarithmic singularity at $\rho=0$. It is remarkable that the analytic property of this function around $\rho=0$ is given before calculating it from (11) explicitly. We can repeat the same argument to see that all functions ${}_{2l+1}f_n$ ($l=0,1,2,\dots$) along the right edge of the diagram are convergent and have no logarithmic singularities at $\rho=0$. Similarly all ${}_{2l}g_n$ ($l=0,1,2,\dots$) have the same properties. Therefore all functions in the right half of the diagram have been shown to be convergent and do not have logarithmic singularities as long as the index of ${}_0s_n^{(2)}$ of the initial function ${}_0g_n$ is an increasing function of n. We cannot say anything about the other half of the diagram since ${}_1f_n$, whose index is ${}_1s_n^{(1)}$ by definition, may not have such properties.

The most important feature of the diagram to get the above result is that the DR ensures that the converging solution of the $(k-2)$th LBT is one of the solutions of the kth LBT with smaller values of indices. We will show later that the Bessel-type case offers a real example of this situation.

Next we consider briefly the behaviours of solutions to the kth LBT equation around $\rho=\rho_0\neq 0$. Let us expand the potential ${}_{k-1}g_n$ and the solution ${}_kf_n$ in power series of $(\rho-\rho_0)$ in the same forms as in the case of (15), (16), and (19). Then we get, for the coefficients of the kth LBT,

$$ {}_kP_n = -\frac{{}_{k-1}s+{}_{k-1}\bar{s}}{\rho-\rho_0}+(\mathrm{FT}) \tag{27}$$

$$ {}_kQ_n = \frac{{}_{k-1}s\cdot{}_{k-1}\bar{s}}{(\rho-\rho_0)^2}+\frac{1}{\rho-\rho_0}\{{}_{k-1}s\cdot(\mathrm{FT})+{}_{k-1}\bar{s}\cdot(\mathrm{FT})\}+(\mathrm{FT}) \tag{28}$$

where ${}_{k-1}s={}_{k-1}s_n$ and ${}_{k-1}\bar{s}={}_{k-1}s_{n+1}$ indicate the lowest powers in $(\rho-\rho_0)$ of ${}_{k-1}g_n$ and ${}_{k-1}g_{n+1}$ respectively and (FT) denotes the finite term(s) at $\rho=\rho_0$.

From the expressions (27) and (28), $\rho=\rho_0\neq 0$ is again a regular singular point, and the indicial equation appears to be

$$ {}_ks_n^2-({}_{k-1}s_n+{}_{k-1}s_{n+1}+1)\,{}_ks_n+{}_{k-1}s_n\cdot{}_{k-1}s_{n+1}=0. \tag{29}$$

Using (29), we can determine the dominant local behaviour of solutions around $\rho=\rho_0\neq 0$. First we notice that two solutions ${}_ks_n^{(1)}$ and ${}_ks_n^{(2)}$ of (29) satisfy

$$\begin{aligned} &{}_ks_n^{(1)}+{}_ks_n^{(2)}={}_{k-1}s_n+{}_{k-1}s_{n+1}+1\\ &{}_ks_n^{(1)}\cdot{}_ks_n^{(2)}={}_{k-1}s_n\cdot{}_{k-1}s_{n+1}. \end{aligned}\tag{30}$$

From these equations we can draw some immediate results.

(1) ${}_k s_n^{(1)}$ and ${}_k s_n^{(2)}$ are the same (opposite) sign if ${}_{k-1}s_n$ and ${}_{k-1}s_{n+1}$ are the same (opposite) sign.

(2) If ${}_{k-1}s_n$ (${}_{k-1}s_{n+1}$) is zero, then one of ${}_k s_n$, say ${}_k s_n^{(1)}$, vanishes and the other one, ${}_k s_n^{(2)}$, is given by ${}_{k-1}s_{n+1}+1$ (${}_{k-1}s_n+1$). For example, if ${}_{k-1}g_n$ has a simple zero and ${}_{k-1}g_{n+1}$ is a constant at $\rho=\rho_0$, one of the solutions has a double zero whereas the other one is a constant.

(3) If both ${}_{k-1}g_n$ and ${}_{k-1}g_{n+1}$ are finite at $\rho=\rho_0$, one of the solutions has a simple zero whereas the other one is a constant.

(4) If ${}_k s_n^{(2)} > {}_k s_n^{(1)}$, (30) shows that

$$|{}_k s_n^{(2)}| > |\max({}_{k-1}s_n, {}_{k-1}s_{n+1})|$$

and

$$|{}_k s_n^{(1)}| < |\min({}_{k-1}s_n, {}_{k-1}s_{n+1})|.$$

(5) ${}_k s_n^{(2)} = {}_k s_n^{(1)}$ only if ${}_{k-1}s_n < 0$, ${}_{k-1}s_{n+1} < 0$.

We now discuss the behaviour of solutions of the LBT at $\rho=\infty$. We start by setting forms of expansion of the gauge potential g_n and the solution f_n at $\rho=\infty$, in the same manner as the previous arguments. Let us give the form of potential as

$$ {}_{k-1}g_n(\rho) = \rho^{{}_{k-1}\sigma_n} + G_n^\infty(p) \tag{31}$$

in which $G_n^\infty(\rho)$ is a finite expansion of ρ^{-1}. Then the coefficients of the LBT are given by

$$ {}_k P_n(\rho) = \rho^{-1}[-({}_k a_n + {}_k b_n - 1 + {}_{k-1}\sigma_n + {}_{k-1}\sigma_{n+1}) + \mathrm{O}(\rho^{-1})] \tag{32}$$

$$ {}_k Q_n(\rho) = \rho^0[1 + ({}_k a_n + {}_{k-1}\sigma_n)({}_k b_n + {}_{k-1}\sigma_{n+1})\rho^{-2} + \mathrm{O}(\rho^{-2})]. \tag{33}$$

We can see directly from this expression that $\rho=\infty$ is a so-called first class of irregular singular points. Then the indicial equation of this point gives values to the indices

$$ {}_k\sigma_n = \pm \mathrm{i}. \tag{34}$$

Following the general method to construct a formal power series solution which gives an asymptotic expansion at infinity, we get a solution which behaves dominantly in the following form:

$$ \begin{aligned} {}_k f_n &\sim \mathrm{e}^{\mp \mathrm{i}\rho}\rho^{1/2({}_k\sigma_n^{(1)} + {}_k\sigma_n^{(2)} - 1)}[1+\mathrm{O}(\rho^{-1})] \\ &= \mathrm{e}^{\mp \mathrm{i}\rho}\rho^{-1/2}[1+\mathrm{O}(\rho^{-1})]. \end{aligned} \tag{35}$$

This behaviour is almost the same as that of Bessel functions.

5.2. *The case of Bessel-type solutions*

Taking Bessel-type solutions as examples, we will show the local behaviour of solutions to the LBT equation around singular points concretely. Further, based on analytical studies, we will present results of the numerical calculation.

First of all, three vertices of the top triangle in figure 3 are given as ${}_0 g_n = 1$, ${}_1 f_n = J_n(\rho)$ and ${}_1\tilde{f}_n = N_n(\rho)$. Since the local behaviours of $J_n(\rho)$ and $N_n(\rho)$ are well studied, we will stop referring to these except to relate the fact that the indicial equation has two solutions ${}_1 s_n^{(1)} = -n$ and ${}_1 s_n^{(2)} = n$ around $\rho = 0$.

In the case of the second LBT, three vertices of the triangle are obtained as ${}_1f_n = J_n(\rho)$, ${}_0g_{n+1} = 1$ and ${}_2g_n = \int J_nJ_{n+1}\,\mathrm{d}\rho$. A linear combination of ${}_0g_{n+1}$ and ${}_2g_n$ is nothing but Nakamura's solution $(1+\varepsilon^2\sum_{m=n}^{\infty} J_m^2(\rho))$ [5] and it is known to be positive definite. We will reconsider these solutions from the viewpoint of a power series expansion.

From the forms of coefficients in the LBT equation, there seem to appear singularities at $\rho = 0$ and zero points of $J_n(\rho)$, all of which are regular singular points. Around $\rho = 0$, we obtain ${}_2s_n^{(1)} = 0$ and ${}_2s_n^{(2)} = 2n+2$ by taking ${}_2a_n = -n$, ${}_2b_n = n$, and ${}_1s_n^{(2)} = n$ in (22′). If we take account of the DR, it is known that one solution having the index ${}_2s_n^{(1)}$ is ${}_0g_{n+1}$ and the other having ${}_2s_n^{(2)}$ is ${}_2g_n$. Both solutions therefore have no logarithmic singularities around $\rho = 0$. At the zero points of $J_n(\rho)$, the indices become constants as ${}_2s_n^{(1)} = 0$ and ${}_2s_n^{(2)} = 2$ and the solutions have finite values at these points. Further, when ρ goes to infinity, two solutions also converge as

$$\int_0^{\infty} J_nJ_{n+1}\,\mathrm{d}\rho = \tfrac{1}{2}.$$

Figure 4 shows the result of numerical calculation of Nakamura's solution with $\varepsilon^2 = 1$ for $0 \leqslant \rho \leqslant 20$ and for $0 \leqslant n \leqslant 4$.

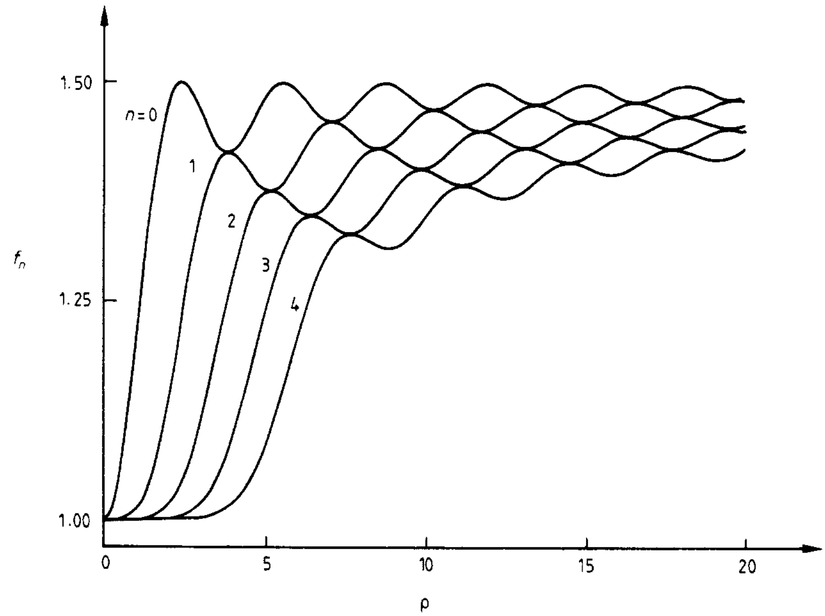

Figure 4. Solutions of the second LBT equation: $f_n(\rho) = 1 + \int J_nJ_{n+1}\,\mathrm{d}\rho$ for $0 \leqslant \rho \leqslant 20$ and $0 \leqslant n \leqslant 4$.

At the next level of the LBT, we find a new solution expressed by an indefinite integral, whose local behaviour is not known immediately. To see it, it is effective to expand the solution around the singular point. According to the discussion in § 5.1, two indices are obtained as ${}_3s_n^{(1)} = n+1$ and ${}_3s_n^{(2)} = 3n+5$ when we put ${}_3a_n = -(n+1)$, ${}_3b_n = n+1$ and ${}_2s_n^{(2)} = 2n+2$ into (22′). One solution having the index ${}_3s_n^{(1)}$ is $J_{n+1}(\rho)$ and the other is $J_{n+1}\cdot\int(\int J_nJ_{n+1}\,\mathrm{d}\rho)(\int J_{n+1}J_{n+2}\,\mathrm{d}\rho)/(\rho J_{n+1}^2)\,\mathrm{d}\rho$ in figure 3. Also at the zero points of $J_{n+1}(\rho)$, the latter solution has finite values. Figure 5 shows the result of numerical calculation of the latter solution for $0 \leqslant \rho \leqslant 20$ and for $0 \leqslant n \leqslant 4$.

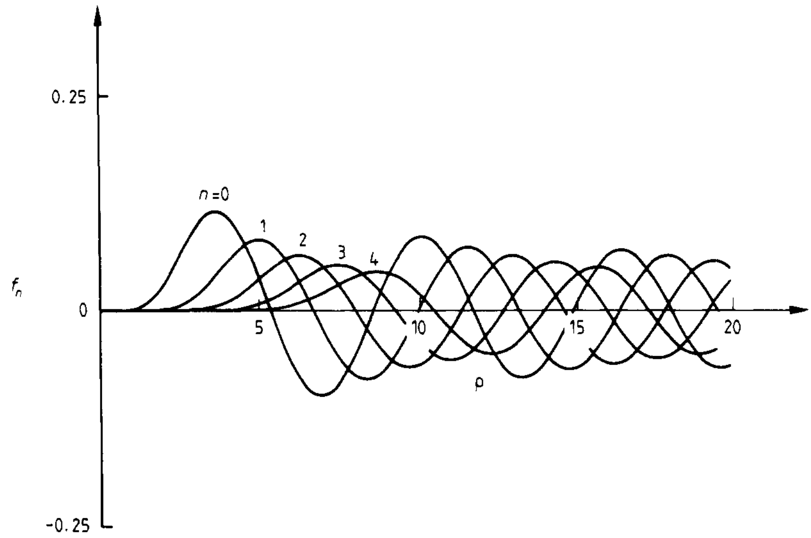

Figure 5. Solutions of the third LBT equation:

$$f_n(\rho) = J_{n+1}(\rho) \cdot \int \frac{\int J_n J_{n+1}\,\mathrm{d}\rho \cdot \int J_{n+1} J_{n+2}\,\mathrm{d}\rho}{\rho J_{n+1}^2}\,\mathrm{d}\rho$$

for $0 \leq \rho \leq 20$ and $0 \leq n \leq 4$.

In a similar manner, we get two indices around $\rho = 0$, namely ${}_4s_n^{(1)} = 2n+4$ and ${}_4s_n^{(2)} = 4n+10$, taking ${}_4a_n = -(n+1)$, ${}_4b_n = n+2$, and ${}_3s_n^{(2)} = 3n+5$ in (22'). Also in this case, it is known that neither solution has a logarithmic singularity around $\rho = 0$. The result of numerical calculation of a linear combination of the two solutions is presented in figure 6 for $0 \leq \rho \leq 20$ and for $0 \leq n \leq 3$.

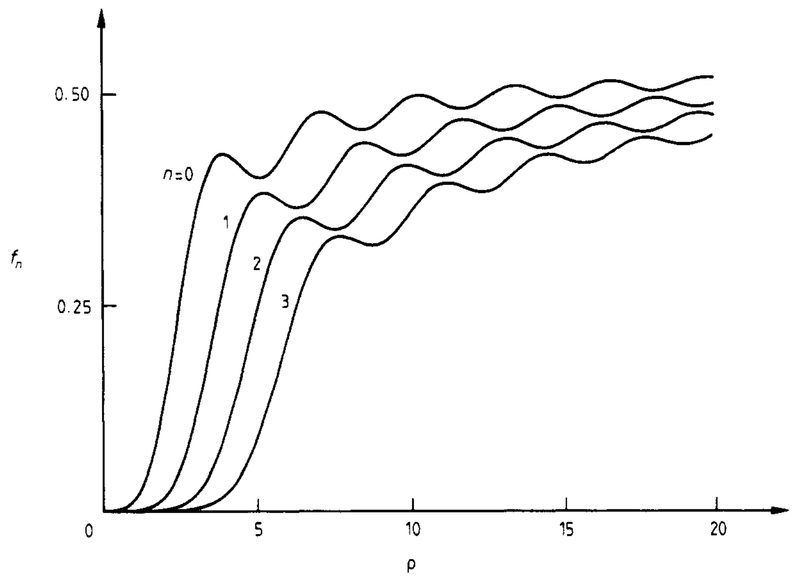

Figure 6. Solutions of the fourth LBT equation for $0 \leq \rho \leq 20$ and $0 \leq n \leq 3$.

6. Discussion

In the previous sections, we discussed the duality relation and the linear Bäcklund transformation in the case of the cylindrically symmetric Toda lattice equation, and constructed a diagram of solutions generated by the LBT and the DR. We defined a unit diagram in § 3 corresponding to a fundamental set of solutions to the kth-level LBT equation. One of the fundamental solutions is simply given by a solution of the $(k-2)$th-level LBT equation as a result of the DR. Another solution independent of the former is obtained through the standard procedure to solve linear differential equations of second order.

The essential point of this construction, which generates solutions to the 2DTL by successive linear equations, is the duality of gauge fields and matter fields. The point has been clarified in the discussions of § 2.

Based on the discussions and conclusions of §§ 2 and 3, we constructed the whole diagram of the LBT and found new solutions successively. In the process of obtaining the new solutions, only the DR and knowledge of solving ordinary differential linear equations have been used. As a by-product, we made it clear that Nakamura's Bessel-type solutions are naturally included as solutions produced in our diagram.

We furthermore discussed, in § 5, behaviour of solutions to the cylindrically symmetric 2DTL equation, especially around singular points of the LBT equation associated with singularities of potentials. We can analyse the behaviour of the 2DTL equations entirely based on linear theory through the LBT and the DR.

Among the conclusions discussed in § 5, one of the most interesting and important facts is that the DR gives a solution to the kth level of the LBT equation which has no logarithmic singularities anywhere. This enables us to determine local behaviours of the solutions entirely. According to the conclusions, the singularities of the LBT equation are always of regular type except at $\rho=\infty$. Moreover, if we assume that solutions of the first and second level of the LBT equation have integral indices of their local power series expansions around any point of the independent variable, all the solutions produced in that half of the diagram have no singularities as long as ρ is finite. The Bessel-type solutions provide such examples.

Concerning the behaviour of solutions it is very interesting that the infinite point $\rho=\infty$ appears to be an irregular singular point in the LBT equation (8) with (9) and (10). It is also very interesting that we can get rid of this irregularity, however, by a slight modification of the cylindrically symmetric 2DTL equation, as we will discuss briefly in the following.

The irregularity of the singular point of (8) at $\rho=\infty$ comes from the first term of ${}_kQ_n$ in (10). As we will see, this term corresponds to a particular choice of the boundary condition at $\rho=\infty$. To clarify this point let us recall the discussions of [15] where the 2DTL was derived as a compatibility condition of the DR in Cartesian coordinates. There the DR are given by

$$f_{n\pm 1}(x, y) = c_{\mp}\left(\frac{g_{n+1}}{g_n}\right)^{\pm 1} \nabla_{\mp} f_n(x, y) \tag{36}$$

where

$$\begin{aligned} \nabla_+ &= \partial_+ - (\partial_+ \ln g_{n+1}) \\ \nabla_- &= \partial_- - (\partial_- \ln g_n) \\ \partial_\pm &= \partial_x \pm \mathrm{i}\partial_y \end{aligned} \tag{37}$$

and $c_\pm$ are arbitrary constants. Equations (35) are equivalent to Hirota's bilinear Bäcklund transformation in [11]. We have similar equations in which the roles of g_n and f_n are interchanged. The compatibility conditions of two equations, equation (36), is then

$$[\nabla_+, \nabla_-] = \partial_+\partial_- \ln \frac{g_{n+1}}{g_n} \tag{38}$$

or, writing this explicitly,

$$\partial_+\partial_- \ln g_{n+1} - \partial_+\partial_- \ln g_n = c_+c_-\left(\frac{g_n g_{n+2}}{g_{n+1}^2} - \frac{g_{n-1}g_{n+1}}{g_n^2}\right). \tag{38'}$$

From this equation we get a bilinear form of the 2DTL equation

$$\partial_+\partial_- \ln g_n + c_+c_- \frac{g_{n-1}g_{n+1}}{g_n^2} = G(x, y) \tag{39}$$

where $G(x, y)$ is an n-independent function. When a potential g_n satisfies the bilinear equation (38), the solution f_n of (36) is obtained by solving the single linear differential equation

$$\begin{aligned}\partial_+\partial_- f_n - (\partial_+ \ln g_{n+1})(\partial_- f_n) - (\partial_- \ln g_n)(\partial_+ f_n)\\ + [G(x, y) + (\partial_+ \ln g_{n+1})(\partial_- \ln g_n)] f_n = 0\end{aligned} \tag{40}$$

where we have fixed $c_+ = c_- = 1$. It is clear from this expression that our equation (8) corresponds to $G(x, y) = 1$. If we had chosen $G(x, y) = 0$, we would have had $\rho = \infty$ as a regular singular point associated with a different boundary condition. All the other singular points, including $\rho = 0$, remain regular, hence the LBT equation turns out to be Fuchsian. Corresponding to $G(x, y) = 0$, we will be able to draw another set of diagrams whose elements are characterised by a different boundary condition.

Acknowledgment

One of the authors (NS) was partially supported by the Japanese Ministry of Education.

References

[1] Toda M 1967 *J. Phys. Soc. Japan* **22** 431
[2] Toda M 1981 *Theory of Nonlinear Lattices* (Berlin: Springer)
[3] Mikhailov A V 1979 *JETP Lett.* **30** 414
[4] Ueno K and Takasaki K 1984 *Adv. Stud. Pure Math.* **4** 1
[5] Nakamura A 1983 *J. Phys. Soc. Japan* **52** 380
[6] Nakamura A 1981 *J. Math. Phys.* **22** 2456
[7] Hirota R 1976 *Bäcklund Transformations, Inverse Scattering Method, Solitons and their Applications* ed R M Miura (Lecture Notes in Mathematics **575**) (Berlin: Springer) p 40
[8] Kametaka Y 1984 *Proc. Jap. Acad.* Ser A **60** 121, 181
[9] Saitoh N, Takizawa E I and Takeno S 1985 *J. Phys. Soc. Japan* **54** 4524
[10] Saitoh N 1986 *J. Phys. Soc. Japan* **55** 435
[11] Hirota R 1981 *J. Phys. Soc. Japan* **50** 3785
[12] Saitoh N and Saito S 1986 *Phys. Lett.* **119A** 287
[13] Saito S and Saitoh N 1987 *J. Math. Phys.* **28** 1052

[14] Saito S and Saitoh N 1987 *Phys. Lett.* **120A** 322
[15] Yuasa F, Saitoh N and Takizawa É I 1987 *J. Phys. Soc. Japan* **54** 3813
[16] Maxon S and Viecelli J 1974 *Phys. Fluids* **17** 1614
[17] Saitoh N and Saito S 1987 *J. Phys. Soc. Japan* **56** 1664
[18] Saitoh N and Takizawa É I 1986 *J. Phys. Soc. Japan* **55** 1827
[19] Yuasa F and Takizawa É I 1987 *J. Phys. Soc. Japan* **56** 3740